1. Definitions in Graph Theory

Definition 1. *A* ***graph*** $G = (V_G, E_G)$ *is an ordered pair the vertex set* V_G *and the edge set* $E_G \subseteq V_G \times V_G$ *such that*

- *For all* $v \in V_G$, $(v, v) \notin E$.
- *For all* $v, w \in V_G$, $(v, w) \in E_G$ *if and only if* $(w, v) \in E_G$.

When there is no ambiguity regarding vertex and edge sets, we will refer to V_G *simply as* V *and* E_G *as* E*. When discussing edges (elements of the edge set), we will identify* (v, w) *with* (w, v)*. By convention, we may write an edge* $e = (v, w)$ *(where* $v, w \in V$*) as* vw *or, equivalently,* wv.

Definition 2. *Given a vertex* $v \in V$ *of graph* G*, the* ***degree*** *of* v *is the number of edges connecting to* v*, i.e.*

$$deg(v) := \#\left(\{(v, w) \in E \mid w \in V\}\right) = \#\left(\{(w, v) \in E \mid w \in V\}\right).$$

We may then determine that $2\#(E) = \sum_{v \in V} \deg(V)$.

Definition 3. *Given a graph* G*, a* ***path*** (v_i) ***in*** G ***of length*** n *is a sequence of distinct vertices* $v_1, \ldots, v_{n+1}$ *such that* $v_i v_{i+1} \in E$ *for each* $i \in \{1, \ldots, n\}$*. A* ***cycle in*** G ***of length*** $n \geq 3$ *is a sequence of vertices* $v_1, \ldots, v_{n+1}$ *such that* $(v_1, \ldots, v_n)$ *is a path,* $v_n v_{n+1} \in E$*, and* $v_{n+1} = v_1$.

One may note, for instance, that if n is the minimal degree of vertices in G, then there is a path of length at least n.

Definition 4. *A graph* G *is* ***connected*** *if for any two vertices* v *and* w*, there exists a path in* G *beginning at* v *and ending at* w.

We now have all the basic tools of graph theory and may now proceed to formalize these notions into some algebraic setting. Our algebraic object of choice will be a vector space, and in particular, we will consider vector spaces over $\mathbb{F}_2$, the finite field of 2 elements.

Definition 5. *Given a graph* G*, the* ***edge space*** $\mathcal{E}$ *is the free vector space over* $\mathbb{F}_2$ *generated by* E*. Elements of* $\mathcal{E}$ *correspond to subsets of* G*, and the vector addition corresponds to the symmetric difference.*

Definition 6. *Given a graph* G*, the* ***cycle space*** $\mathcal{C}$ *is the subspace of* $\mathcal{E}$ *spanned by all the elements of* $\mathcal{E}$ *corresponding to cycles in* G.

Theorem 1. *A subset* S *of the edge set corresponds to an element of the cycle space* $\mathcal{C}$ *if and only if each vertex of the subgraph determined by* S *has even degree.*